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On angular bisectors in normed linear spaces

Horst Martini
Faculty of Mathematics, Chemnitz University of Technology
horst.martini@mathematik.tu-chemnitz.de

Senlin Wu[i]
Department of Applied Mathematics, Harbin University of Science and Technology
senlin_wu@hotmail.com

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Abstract. We prove that a Minkowski space is Euclidean if it has the weak bisector property. This confirms a conjecture of R. W. Freese, C. R. Diminnie, and E. Z. Andalafte.

Keywords: angular bisector, angular measure, characterization of inner product spaces, Minkowski plane, Minkowski space, norm, weak bisector property

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By X we denote the *Minkowski space* (i.e., the finite dimensional real Banach space) with *origin* o, *norm* $\|\cdot\|$, and *unit sphere* $S_X := \{x \in X : \ \|x\| = 1\}$. When X is a Minkowski plane, i.e., a two-dimensional Minkowski space, S_X is also called the *unit circle* of X. Basic references for the geometry of Minkowski spaces are [5], [4], and the monograph [8]. For $x \neq -y$, the intersection of the cone $\{\lambda x + \mu y : \ \lambda, \mu \geq 0\}$ and S_X is called the *(undirected) arc* between x and y and denoted by $S_X(x, y)$, and the length of $S_X(x, y)$ is denoted by $\delta_X(x, y)$. For brevity, we use the shorthand notation $\hat{x} = \frac{x}{\|x\|}$ for any point $x \neq o$.

For any three non-collinear points p, x, $y \in X$ we call the convex set bounded by the rays $[p, x\rangle$ and $[p, y\rangle$ the *angle* xpy (denoted by $\angle xpy$) with p as apex. For two linearly independent points x, $y \in S_X$ the authors of [2] defined the measure of $\angle xoy$ by

$$A(x, y) := \cos^{-1}\left[\frac{1}{2}(2 - \|\hat{x} - \hat{y}\|^2)\right].$$

The ray $[o, z\rangle$ with $z \in \angle xoy$ is called the *angular bisector* of $\angle xoy$ provided $A(x, z) = A(y, z)$ (the uniqueness of z in this framework follows from [3, p. 170, Corollary]). R. W. Freese, C. R. Diminnie, and E. Z. Andalafte in [3] proved that a Minkowski space X is Euclidean if X has the *angle bisector property*, where X is said to have the angle bisector property if for all linearly independent x, $y \in S_X$ the point $z = \widehat{x+y}$ satisfies $A(x, z) = A(y, z) = \frac{1}{2}A(x, y)$. In [6] a stronger result was proved (see Theorem 4.2 there): If for all linearly independent x, $y \in S_X$ the point $z = \widehat{x+y}$ satisfies $A(x, z) = A(y, z)$, then the underlying Minkowski plane is Euclidean. The authors of [3] also asked whether the *weak bisector property*, which is formulated in the following, still implies that the underlying space is Euclidean.

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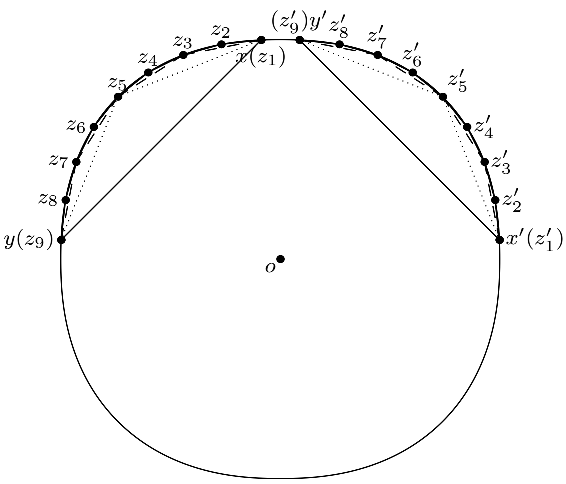

Figure 1. Proof of Theorem 1.

Weak bisector property: *For any two linearly independent points x, $y \in S_X$ there exists a point $z \in \angle xoy$ having the property $A(x, z) = A(y, z)$ which satisfies $A(x, z) = \frac{1}{2}A(x, y)$.*

Once more we mention that in view of [3, p. 170, Corollary] this point z is unique, up to positive multiples. The aim of this paper is to give an affirmative answer to this question, namely by

Theorem 1. *A Minkowski space X of dimension at least 2 is Euclidean if it has the weak bisector property.*

To prove this result we need the following lemma:

Lemma 1 (See Corollary 2.5 in [7]). *Let X be a Minkowski plane. If there exists a function $\varphi : [0, 2] \to [0, 4]$ such that for any $u, v \in S_X$ we have $\delta_X(u, v) = \varphi(\|u - v\|)$, then X is Euclidean.*

Proof of Theorem 1: Since a Minkowski space of dimension at least 2 is Euclidean if and only if each of its two-dimensional subspaces is Euclidean (see [1]), we may assume, without loss of generality, that X is a Minkowski plane. By Lemma 2, we only need to show that the length of any undirected arc of S_X is determined by the length of the corresponding chord.

Note first that for any x, $y \in S_X$ there exists a unique point $z \in S_X$ with $z \in \angle xoy$ and $A(x, z) = A(y, z)$, and that $\|x - z\| = \|y - z\|$ is determined only by $\|x - y\|$. Indeed, from the assumption of the theorem it follows that

$$\cos^{-1}\left[\frac{1}{2}(2 - \|\hat{x} - \hat{z}\|^2)\right] = A(x, z) = \frac{1}{2}A(x, y) = \frac{1}{2}\cos^{-1}\left[\frac{1}{2}(2 - \|\hat{x} - \hat{y}\|^2)\right],$$

which implies that

$$\|\hat{x} - \hat{z}\| = \sqrt{2 - \sqrt{4 - \|\hat{x} - \hat{y}\|^2}}.$$

Now, for any points x, y, x', $y' \in S_X$ with $\|x - y\| = \|x' - y'\|$, we show that $\delta_X(x, y) = \delta_X(x', y')$. For any integer $n \geq 1$ we have two subsets $\{z_1, \cdots z_{2^n+1}\}$, $\{z_1', \cdots z_{2^n+1}'\} \subset S_X$

such that the following holds (note that the points z_i etc. are found again in view of [3, p. 170, Corollary]):

(1) $\{z_1, \cdots z_{2^n+1}\} \subset S_X(x,y)$ and $\{z'_1, \cdots z'_{2^n+1}\} \subset S_X(x',y')$,

(2) $A(z_1, z_{2^{n-1}+1}) = A(z_{2^n+1}, z_{2^{n-1}+1}) = A(z'_1, z'_{2^{n-1}+1}) = A(z'_{2^n+1}, z'_{2^{n-1}+1})$,

(3)

$$\begin{aligned} A(z_1, z_{2^{n-2}+1}) &= A(z_{2^{n-2}+1}, z_{2^{n-1}+1}) \\ &= A(z'_1, z'_{2^{n-2}+1}) = A(z'_{2^{n-2}+1}, z'_{2^{n-1}+1}) \end{aligned}$$

and

$$\begin{aligned} A(z_{2^{n-1}+1}, z_{2^{n-1}+2^{n-2}+1}) &= A(z_{2^n+1}, z_{2^{n-1}+2^{n-2}+1}) \\ &= A(z'_{2^{n-1}+1}, z'_{2^{n-1}+2^{n-2}+1}) \\ &= A(z'_{2^n+1}, z'_{2^{n-1}+2^{n-2}+1}) \end{aligned}$$

...

Continuing with similar arguments, we finally obtain that

$$\|z_i - z_{i+1}\| = \|z_j - z_{j+1}\| = \|z'_i - z'_{i+1}\| = \|z'_j - z'_{j+1}\|$$

holds for any $1 \le i, j \le 2^n$.

Hence

$$\delta_X(x,y) = \sup_{n\ge 1} \left\{ \sum_{i=1}^{2^n} \|z_i - z_{i+1}\| \right\} = \sup_{n\ge 1} \left\{ \sum_{i=1}^{2^n} \|z'_i - z'_{i+1}\| \right\} = \delta_X(x',y'),$$

and this supremum of sums converges to the arc length since $\|z_i - z_{i+1}\|$ converge to zero. This follows since that sum is bounded from above by the arc length between x and y, and we have 2^n summands for $n \to \infty$. The proof is complete. $\boxed{QED}$

Remark 1. The weak bisector property in Theorem 1 can be replaced by the following condition: There exists a real function ϕ such that for any two linearly independent points x, $y \in S_X$, the point $z \in \angle xoy$ having the property $A(x,z) = A(y,z)$ satisfies $A(x,z) = \phi(A(x,y))$.

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